MATLAB-IMPLEMENTATION OF M2A2-ALGORITHM FOR OPTIMAL
SINGULAR ADAPTIVE OBSERVATION OF SISO LINEAR DISCRETE
SYSTEMS

Assoc.Prof., Dr. Lyubomir Nikolaev Sotirov
Ass.Prof. Nikola Nikolaev Nikolov
Technical University of Varna, Bulgaria
Faculty of Computer Science and Automation
E-mail: Sotirov@ms.ieee.bg

Abstract. MATLAB-implementation of an algorithm for synthesis of the new
generation of discrete adaptive observers, including optimal estimators of the initial state
vector, identifiers of parameters and discrete optimal singular (full and degenerate) adaptive
observers of the current state vector, for SISO linear discrete systems is suggested. The
estimation of the initial state vector elements is not typical for the methods for synthesis of
discrete adaptive observers, based only on identifiers. The synthesized M2A2-algorithm has
features such as large-scale granularity and natural parallelism, which also allow its
software and hardware implementation on a parallel computer architecture.

Keywords: optimal singular adaptive observation, identification, initial state vector
estimation, discrete systems, MATLAB-implementation.

1. Introduction

Observed discrete single-input single-output (SISO) systems of the following type:

\[(1) \quad x(k+1) = Ax(k) + bu(k), \quad x(0) = x_0,\]
\[(2) \quad y(k) = c^T x(k), \quad k = 0, 1, 2, ..., \]

are considered, where \(x(k) \in \mathbb{R}^n\) is an unknown current state vector, \(x(0) \in \mathbb{R}^n\) is an unknown
initial state vector, \(u(k) \in \mathbb{R}^1\) is a scalar input, \(y(k) \in \mathbb{R}^1\) is a scalar output, \(A\) and \(b\) are
unknown matrix and a vector of the following type:

\[(3) \quad A = \begin{bmatrix} 0 & \cdots & I_{n-1} \\ \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad a^T = [a_1, a_2, \cdots, a_n],\]

where \(I_{n-1}\) is an \((n-1) \times (n-1)\) identity matrix;

\[(4) \quad b^T = [b_1, b_2, \cdots, b_n];\]
\[(5) \quad c^T = [1, 0, \cdots, 0].\]

The discrete optimal singular adaptive observation problem is in constructing of the
new generation of discrete adaptive observers, including optimal estimators of the initial state
vector \(x(0)\), identifiers of parameters \(a\) and \(b\), and discrete optimal singular (full and
degenerate) adaptive observers of the current state vector \(x(k), k = 1, 2, ...\)
The synthesis of the M2A2-algorithm is carried out with the help of the toolbox of the theory for optimal singular adaptive (OSA) observation.

An impression of the state and development of the theory of optimal singular adaptive observation could be obtained from (Sotirov 1994-1999).

In this paper we implement the M2A2-algorithm in MATLAB environment.

2. MATLAB-implementation of the M2A2-algorithm for OSA observation

Step 1. Vectors $y_1$, $y_2$, $y_3$ and matrices of input-output data are formed:

```matlab
for i=1:n
    y1=[y(i)];
end
for i=n+1:2*n
    y2=[y(i)];
end
for i=2*n+1:3*n
    y3=[y(i)];
end
for j=1:n-1
    for q=1:n-j
        i=q+j;
        U(i,j)=u(q);
    end
end
U11=[U,zeros(n,1)];
for i=1:n
    for m=1:i+n-1
        j=m-n+1;
        U11(i,j)=u(m);
    end
end
```

where $U11$, $U21$, and $U31$ are $n \times n$ Toeplitz matrices; $Y22$ and $Y32$ are $n \times n$ Hankel matrices.

**Step 2.** The estimate $h0$ of the vector

$$h^T = [h_1, h_2, \ldots, h_n]$$

is calculated as follows:

$$N1=U21-Y22*Y32^\times-1*U31;$$

$$h0=N1^\times-1*y2-N1^\times-1*Y22*Y32^\times-1*y3;$$

**Step 3.** The estimate $ao$ of the vector $a$ is calculated with the vector-matrix expression of the kind:

$$ao=-Y32^\times-1*U31*N1^\times-1*y2+Y32^\times-1+Y32^\times-1*U31*N1^\times-1*Y22*Y32^\times-1*y3;$$

**Step 4.** The estimate $eta00$ of the vector

$$\eta^T(0) = [\eta_1(0), \eta_2(0), \ldots, \eta_n(0)]$$

is calculated with the following vector-matrix expression:

$$eta00=y1-U11*N1^\times-1*y2+U11*N1^\times-1*Y22*Y32^\times-1*y3;$$

**Step 5.** The estimates of the elements of the vector $b$ are recursively calculated with the following module:

```matlab
bo(1)=ho(1);
for i=2:n
    S=0;
```
for j=1:i-1
    S=S+ao(n+1-i+j)*bo(j);
    bo(i)=ho(i)+S;
end

bo=bo';

Step 6. The initial state vector estimate \( x_0 \) is calculated, using the optimal estimator of the form:

\[
x_0 = \eta_0 + U_11^* (\eta_0 - \eta_0')
\]

Step 7. The current state vector \( x_0(:, k) \) is estimated, using the following discrete full optimal singular adaptive observer:

\[
I = \text{eye}(n-1);
NUL = \text{zeros}(n-1,1);
Ao = [NUL, I; ao'];
Fo = Ao - g'*c';
xo(:,1) = xo0;
for k=1:3*n-1
    xo(:,k+1) = Fo*xo(:,k) + bo*u(k,:) + g*xo(1,k);
end
\]

where the choice of the \( g \)-vector elements is arbitrary and is not hindered by the solution of the problem of the synthesis of systems with given poles, as it is in the case with Luenberger’s non-adaptive observers and also might be done, e.g. in one of the following ways:

1) \( g = 0 \);
2) \( g = -bo' \);
3) to choose \( g \) in such a way, that the matrix \( Fo \) has eigenvalues, situated in the single circle.

The discrete degenerate optimal singular adaptive observer (at \( g = 0 \)) might be regarded as a degenerate case of the full OSA observer and might be constructed easier.

The following theorem is a consequence of the M2A2-algorithm:

**Theorem M2A2:** The unique solution of the given discrete optimal singular adaptive observation problem, using the suggested algorithm, exists if and only if the following conditions are satisfied:

\[
\begin{align*}
\det(Y_{32}) & \neq 0 , \\
\det(U_{21} - Y_{22}^* Y_{32}^{-1} U_{31}) & \neq 0 
\end{align*}
\]


On the base of simulated data, for example, for a discrete system of fourth order, which parameters and initial states are:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-0.75 & 0.97 & -0.3 & 1.2 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
1.5 \\
1.7 \\
0.9 \\
1 \\
\end{bmatrix}, \quad x(0) = \begin{bmatrix}
1.5 \\
1.7 \\
1.9 \\
1.1 \\
\end{bmatrix}
\]
using the following module for simulation of the output signal:

\[
\begin{align*}
    & u = [0.1, 0.2, 0.3, 0.5, 0.4, 0.3, 0, -0.2, -0.3, -0.4, -0.5, -0.3]^T; \\
    & n = 4; \\
    & a = [-0.75; 0.97; -0.3; 1.2]; \\
    & I = \text{eye}(n-1); \\
    & NUL = \text{zeros}(n-1,1); \\
    & A = [NUL, I; a']^T; \\
    & b = [1.5; 1.7; 0.9; 1]; \\
    & x0 = [1.5; 1.7; 1.9; 1.1]; \\
    & c = [1; \text{zeros}(n-1,1)]; \\
    & x = \text{ltitr}(A, b, u, x0)^T; \\
    & k = 1:3*n; \\
    & y(k) = c' * x(:,k); \\
\end{align*}
\]

then

\[
\]

In result, estimates with the following relative errors:

\[
\begin{align*}
    & \frac{\|a - a_0\|}{\|a\|} = 1.3448 \cdot 10^{-14}, \\
    & \frac{\|b - b_0\|}{\|b\|} = 2.0052 \cdot 10^{-14},
\end{align*}
\]

have been obtained, using the identifier.

By that

\[
    \text{cond}(Y2) = 43.7778, \\
    \text{cond}(N) = 17.0797
\]

and

\[
    h_0 = [1.5, -0.1, -0.69, -1.025]^T.
\]

With the help of the optimal estimator of the initial state, estimates with the following relative error are obtained:

\[
    \frac{\|x_0 - x_0^0\|}{\|x_0\|} = 0.0026 \cdot 10^{-12}.
\]

The degenerate OSA observer estimates the current state vector with the following relative errors:

\[
\begin{align*}
    & \frac{\|e(1)\|}{\|x(1)\|} = 0.0087 \cdot 10^{-12}, \\
    & \frac{\|e(2)\|}{\|x(2)\|} = 0.0149 \cdot 10^{-12}, \\
    & \frac{\|e(3)\|}{\|x(3)\|} = 0.0217 \cdot 10^{-12}, \\
    & \ldots, \\
    & \frac{\|e(10)\|}{\|x(10)\|} = 0.1009 \cdot 10^{-12}, \\
    & \frac{\|e(11)\|}{\|x(11)\|} = 0.1183 \cdot 10^{-12}
\end{align*}
\]

where \( e(k) = x(k) - x_0(k), \quad k = 1, 2, \ldots \)

Its poles are the following:

\[
    \text{eig}(F_0) = [0.7516, 1.1873, -0.3695 + 0.8390i, -0.3695 - 0.8390i].
\]

If a full OSA observer (at \( g = b_0 \)) with the following poles:

\[
    \text{eig}(F_0) = [-0.1628, 1.0992, -0.6182 + 1.0745i, -0.6182 - 1.0745i]
\]
is used, the order of estimation stays the same:
\[
\frac{\|e(1)\|}{\|x(1)\|} = 0.0087 \times 10^{-12}, \quad \frac{\|e(2)\|}{\|x(2)\|} = 0.0150 \times 10^{-12}, \quad \frac{\|e(3)\|}{\|x(3)\|} = 0.0219 \times 10^{-12},
\]
\[\ldots, \quad \frac{\|e(10)\|}{\|x(10)\|} = 0.1009 \times 10^{-12}, \quad \frac{\|e(11)\|}{\|x(11)\|} = 0.1183 \times 10^{-12}.\]

The results from the estimation of the full OSA observer at the wished zero poles, i.e. at
\[w^T = [0, 0, 0, 0], \quad g^T = [1.2, 1.14, 1.978, 2.4456]\]
are similar:
\[
\frac{\|e(1)\|}{\|x(1)\|} = 0.0086 \times 10^{-12}, \quad \frac{\|e(2)\|}{\|x(2)\|} = 0.0148 \times 10^{-12}, \quad \frac{\|e(3)\|}{\|x(3)\|} = 0.0216 \times 10^{-12},
\]
\[\ldots, \quad \frac{\|e(10)\|}{\|x(10)\|} = 0.1003 \times 10^{-12}, \quad \frac{\|e(11)\|}{\|x(11)\|} = 0.1177 \times 10^{-12}\]
and its actual poles are the following:
\[\text{eig}(\mathbf{F}_0) = \begin{bmatrix} 0.2370 \times 10^{-3} & -0.2371 \times 10^{-3} & 0.2370i \times 10^{-3} & -0.2370i \times 10^{-3} \end{bmatrix} \]

The characteristics of the third full OSA observer at wished poles of the kind:
\[w^T = [0.1, 0.1, 0.1, 0.1], \quad g^T = [0.8, 0.72, 1.59, 1.7181]\]
are similar to the previous observers, i.e.
\[
\frac{\|e(1)\|}{\|x(1)\|} = 0.0088 \times 10^{-12}, \quad \frac{\|e(2)\|}{\|x(2)\|} = 0.0151 \times 10^{-12}, \quad \frac{\|e(3)\|}{\|x(3)\|} = 0.0221 \times 10^{-12},
\]
\[\ldots, \quad \frac{\|e(10)\|}{\|x(10)\|} = 0.1029 \times 10^{-12}, \quad \frac{\|e(11)\|}{\|x(11)\|} = 0.1205 \times 10^{-12}\]
so that its poles do not coincide with the wished poles, i.e.
\[\text{eig}(\mathbf{F}_0) = \begin{bmatrix} 0.0997, 0.1003, 0.1 + 0.0003i, 0.1 - 0.0003i \end{bmatrix} \]

The relative errors are calculated with the following module:
\[
\text{Na}=\text{norm}(a-a_0) / \text{norm}(a);
\text{Nb}=\text{norm}(b-b_0) / \text{norm}(b);
\text{for } k=1:3*n
\quad \text{N}x(:,k)=\text{norm}(x(:,k)-x_0(:,k)) / \text{norm}(x(:,k));
\end{thefor}
\]

The results of the computing experiments are with highly guaranteed exactness and show the mathematical and software consistency of the constructed algorithm.

4. Conclusion.

The input data for the algorithm are formed as Hankel and Toeplitz matrices, which have some special features. As a result, we may synthesize fast algorithms. They can be used
for the direct inversion of matrices and to solve linear algebraic equation systems with special
structures. This widens the application areas of the proposed method.

Using the suggested algorithm the estimation of the initial state vector elements could
be obtained as well, which is not inherent to the algorithms based on Luenberger’s theory of
observation and to the methods for synthesis of discrete adaptive observers, based only on
identifiers.

The similarity of the observation quality of the four OSA observers, mentioned in the
example, obviously, makes possible the dropping-off of the procedure for synthesis of system
at given poles, hindering the computer application of the results from Luenberger’s
observation theory and the methods for synthesis of adaptive observers, based only on
identifiers.

The algorithm implements a linear computing procedure and avoids the difficulties that
are inherent to non-linear algorithms for parallel estimation of the parameters and the state.

Our algorithm has some features as large-scale granularity and natural parallelism
which provide chances for software and hardware interpretation on a parallel computer
architectures (Sotirov and Sukhov 1997).

References

1. Sotirov L.N., 1994, A direct method for adaptive observation of single-input single-
output linear stationary discrete systems with initial state vector estimation. Reports
of the Bulgarian Academy of Sciences, vol. 47, No.11, 5-8, Sofia, Bulgaria.

2. Sotirov L.N., 1995, A direct method for optimal adaptive observation based on
Luenberger’s identity observer. Automatica and Informatics, No.2, 26-31, Sofia,
Bulgaria.

3. Sotirov L.N., 1996, An algorithm for optimal singular adaptive observation of a class of
discrete systems with initial state vector estimation. Technical Ideas, Scientific Journal
of the Bulgarian Academy of Sciences, No.1, 3-15, Sofia, Bulgaria.

with initial state vector estimation. Automatica and Remote Control, Russian Academy
of Sciences, No.9, 110-118, Moscow, New York, London.

with direct estimation of the initial state vector. Int. J. Systems Science, vol. 28, No.6,
559-562.

estimation of discrete linear system, Automatica and Remote Control, Russian Academy
of Sciences, No.10, 154-163, Moscow, New York, London.

7. Sotirov L.N., 1998, Chosen chapters from the modern control theory, Technical
University of Varna, Bulgaria.

8. Sotirov L.N., 1999, Reduced order optimal singular adaptive observation for a class of
discrete systems, Automatica and Remote Control, Russian Academy of Sciences, No.2,
75-82, Moscow, New York, London.